## Maths for Computing Tutorial 2

**1.** Let S(x) be the predicate "*x* is a student," F(x) the predicate "*x* is a faculty member," and A(x, y) the predicate "*x* has asked *y* a question," where the domain consists of all people associated with your school. Use quantifiers to express each of these statements.

- a) Every student has asked Professor Gross a question.
- b) Every faculty member has either asked Professor Miller a question or been asked a question by Professor Miller.
- c) Some student has not asked any faculty member a question.
- d) There is a faculty member who has never been asked a question by a student. **Solution:** 
  - a)  $\forall x(S(x) \rightarrow A(x, \text{Prof. Gross}))$ . ( $\forall x(S(x) \land A(x, \text{Prof. Gross}))$  is incorrect)
  - b)  $\forall x(F(x) \rightarrow (A(x, \text{Prof. Miller}) \lor A(\text{Prof. Miller}, x))).$
  - c)  $\exists x(S(x) \land \forall y(F(y) \rightarrow \neg A(x, y)))$ .  $(\exists x(S(x) \rightarrow \forall y(F(y) \rightarrow \neg A(x, y)))$  is incorrect) d)  $\exists x(F(x) \land \forall y(S(y) \rightarrow \neg A(y, x)))$ .
- 2. Find a common domain for the variables *x*, *y*, and *z* for which the statement

 $\forall x \forall y ((x \neq y) \rightarrow \forall z ((z = x) \lor (z = y)))$  is true and another domain for which it is false. **Solution:** 

Domains for which the statement is true:  $\{1\}, \{2,3\}$ , or any set of one or two elements. Domains for which the statement is false:  $\{1,2,3\}$ , or any set of three or more elements.

3. Show that the following pairs are not logically equivalent

a) 
$$\exists x P(x) \rightarrow \exists x Q(x) \text{ and } \exists x Q(x) \rightarrow \exists x P(x)$$

b)  $\forall x P(x) \lor \forall x Q(x)$  and  $\forall x (P(x) \lor Q(x))$ 

## Solution:

- a) Take domain as {1} and P(x) = x = 1 and Q(x) = x = 2. Now  $\exists x P(x) \rightarrow \exists x Q(x)$  is false, but  $\exists x Q(x) \rightarrow \exists x P(x)$  is true.
- b) Take domain as set of integers and P(x) = x is even and Q(x) = x is odd. Now  $\forall x P(x) \lor \forall x Q(x)$  is false, but  $\forall x (P(x) \lor Q(x))$  is true.
- 4. Show that the following arguments are valid. (Write all the steps with reasons.)
  - a) Premises:  $p \to (\neg r \to \neg q), \neg r$ . Conclusion:  $\neg (p \land q)$
  - b) Premises:  $p \to q$ ,  $(q \lor r) \land (\neg (q \land r))$ . Conclusion:  $\neg q \to (\neg p \land r)$ .
  - c) Premises:  $p \land \neg s, q \to (r \to s)$ . Conclusion:  $(p \to q) \to \neg r$ .
  - d) Premises:  $\forall x (P(x) \lor Q(x)), \forall x (\neg Q(x) \lor S(x)), \forall x (R(x) \to \neg S(x)), \text{ and } \exists x \neg P(x)$

Conclusion:  $\exists x \neg R(x)$ . (Domain for all quantifiers are the same.)

e) Premises:  $\forall x (P(x) \lor Q(x)), \forall x ((\neg P(x) \land Q(x)) \rightarrow R(x)).$ 

Conclusion:  $\forall x (\neg R(x) \rightarrow P(x))$  (Domain for all quantifiers are the same.) **Solution:** 

- a) 1.  $p \rightarrow (\neg r \rightarrow \neg q)$  (Premise) 2.  $p \rightarrow (r \lor \neg q)$  (Using  $p \rightarrow q \equiv \neg p \lor q$  on 1) 3.  $\neg p \lor (r \lor \neg q)$  (Using  $p \rightarrow q \equiv \neg p \lor q$  on 2) 4.  $\neg p \lor (\neg q \lor r)$  (Commutative law on 3) 5.  $(\neg p \lor \neg q) \lor r$  (Associative law on 4) 6.  $r \lor (\neg p \lor \neg q)$  (Commutative law on 5) 7.  $\neg r$  (Premise) 8.  $\neg p \lor \neg q$  (Disjunctive syllogism on 6 and 7) 9.  $\neg (p \land q)$  (De Morgan's law on 8)
- b) 1.  $p \rightarrow q$  (Premise)

2. 
$$\neg p \lor q$$
 (Using  $p \to q \equiv \neg p \lor q$  on 1)

- 3.  $q \lor \neg p$  (Commutative law on 2)
- 4.  $(q \lor r) \land (\neg (q \land r))$  (Premise)
- 5.  $(q \lor r)$  (Simplification of 4)
- 6.  $(q \lor \neg p) \land (q \lor r)$  (Conjunction of 3 and 5)
- 7.  $q \lor (\neg p \land r)$  (Distributive law on 6)
- 8.  $\neg q \rightarrow (\neg p \land r)$  (Using  $p \rightarrow q \equiv \neg p \lor q$  on 7)
- c) 1.  $p \land \neg s$  (Premise)
  - 2. *p* (Simplification of 1)
  - 3.  $\neg s$  (Simplification of 1)
  - 4.  $q \rightarrow (r \rightarrow s)$  (Premise)
  - 5.  $q \rightarrow (\neg r \lor s)$  (Using  $p \rightarrow q \equiv \neg p \lor q$  on 4)
  - 6.  $\neg q \lor (\neg r \lor s)$  (Using  $p \to q \equiv \neg p \lor q$  on 5)
  - 7.  $(\neg q \lor \neg r) \lor s$  (Associative law on 6)
  - 8.  $s \lor (\neg q \lor \neg r)$  (Commutative law on 6)
  - 9.  $(\neg q \lor \neg r)$  (Disjunctive syllogism on 3 and 8)
  - 10.  $p \land (\neg q \lor \neg r)$  (Conjunction of 2 and 9)
  - 11.  $(p \land \neg q) \lor (p \land \neg r)$  (Distributive law on 10)

12. 
$$(\neg(\neg(p \land \neg q))) \lor (p \land \neg r)$$
 (Double negation on 11)

- 13.  $(\neg (\neg p \lor q)) \lor (p \land \neg r)$  (De Morgan's on 12)
- 14.  $(\neg (p \rightarrow q)) \lor (p \land \neg r)$  (Using  $p \rightarrow q \equiv \neg p \lor q$  on 13)
- 15.  $(\neg (p \rightarrow q) \lor p) \land (\neg (p \rightarrow q) \lor \neg r)$  (Distributive law on 14)
- 16.  $\neg (p \rightarrow q) \lor \neg r$  (Simplification of 15)
- 17.  $(p \to q) \to \neg r$  (Using  $p \to q \equiv \neg p \lor q$  on 16)

d) 1.  $\forall x (P(x) \lor Q(x))$ (premise) 2.  $P(c) \lor Q(c)$  for an arbitrary c in the domain (Universal instantiation of 1) 3.  $\forall x (\neg Q(x) \lor S(x))$ (premise) 4.  $\neg Q(c) \lor S(c)$  for an arbitrary *c* in the domain (Universal instantiation of 3) 5.  $\forall x(R(x) \rightarrow \neg S(x))$  (premise) 6.  $R(c) \rightarrow \neg S(c)$  for an arbitrary c in the domain (Universal instantiation of 5) 7.  $\exists x \neg P(x)$  (premise) 8.  $\neg P(c)$  for some c in the domain (Existential instantiation of 7) 9.  $\neg R(c) \lor \neg S(c)$  for an arbitrary c in the domain (Using  $p \rightarrow q \equiv \neg p \lor q$  on 6) 10.  $\neg Q(c) \lor \neg R(c)$  for an arbitrary *c* in the domain (Resolution on 4 and 9) 11.  $\neg R(c) \lor \neg Q(c)$  for an arbitrary c in the domain (Commutative law on 10) 12.  $P(c) \lor \neg R(c)$  for an arbitrary c in the domain (Resolution on 2 and 11) 13.  $\neg R(c)$  for some c in the domain (Disjunctive Syllogism on 8 and 12) 14.  $\exists x \neg R(x)$ (Existential Generalisation of 13) e) 1.  $\forall x (P(x) \lor Q(x))$ (Premise) 2.  $P(c) \lor Q(c)$  for an arbitrary *c* in the domain (Universal instantiation of 1) 3.  $\forall x((\neg P(x) \land Q(x)) \rightarrow R(x))$  (Premise) 4.  $(\neg P(c) \land Q(c)) \rightarrow R(c)$  for an arbitrary *c* in the domain (Universal instant. of 3) 5.  $\neg(\neg P(c) \land Q(c)) \lor R(c)$  for an arbitrary *c* in ... (Using  $p \rightarrow q \equiv \neg p \lor q$  on 4) 6.  $(P(c) \lor \neg Q(c)) \lor R(c)$  for an arbitrary c in the domain (De Morgan's law on 5) 7.  $(\neg Q(c) \lor P(c)) \lor R(c)$  for an arbitrary c in the domain (Commutative law on 6) 8.  $\neg Q(c) \lor (P(c) \lor R(c))$  for an arbitrary *c* in the domain (Associative law on 7) 9.  $Q(c) \lor P(c)$  for an arbitrary c in the domain (Commutative law on 2) 10.  $P(c) \lor (P(c) \lor R(c))$  for an arbitrary *c* in the domain (Resolution on 8 and 9) 11.  $(P(c) \lor P(c)) \lor R(c)$  for an arbitrary c in the domain (Associative law on 10) 12.  $P(c) \lor R(c)$  for an arbitrary *c* in the domain (Idempotent law on 11) 13.  $R(c) \lor P(c)$  for an arbitrary c in the domain (Commutative law on 12) 14.  $\neg R(c) \rightarrow P(c)$  for an arbitrary c in the domain (Using  $p \rightarrow q \equiv \neg p \lor q$  on 13) 15.  $\forall x (\neg R(x) \rightarrow P(x))$ (Universal generalization of 14)

**5.** Find the flaw in the below proof that shows that if  $\exists x P(x) \land \exists x Q(x)$  is true, then  $\exists x (P(x) \land Q(x))$  is true.

1. $\exists x P(x) \land \exists x Q(x)$	Premise
2. $\exists x P(x)$	Simplification of 1
3. $\exists x Q(x)$	Simplification of 1
4. <i>P</i> ( <i>c</i> )	Existential instantiation from 2
5. $Q(c)$	Existential instantiation from 3

6.  $P(c) \land Q(c)$  Conjunction from 4 and 5 7.  $\exists x (P(x) \land Q(x))$  Existential Generalization **Solution:** We cannot infer  $\exists x (P(x) \land Q(x))$  from  $\exists x P(x) \land \exists x Q(x)$ . Consider the domain as set of integers and P(x) = x is odd and Q(x) = x is even. Then,  $\exists x P(x) \land \exists x Q(x)$  will be true, but  $\exists x (P(x) \land Q(x))$  will be false.

*Flaw:* In 4, it should be "P(c) for some element c in the domain" and in 5, it should be "Q(c) for some element c in the domain". Now, from 4 and 5, we cannot infer  $P(c) \land Q(c)$  as cs in 4 and 5 may not be the same. It is advisable to use different notation when multiple statements of the type  $\exists x P(x)$  are present to avoid confusion. For instance, we could have written "Q(c') for some element c' in the domain".